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Constraints on "rare" dyon decays

Sunil Mukhi and Rahul Nigam

Tata Institute of Fundamental Research,
Homi Bhabha Rd, Mumbai 400 005, India
E-mail: mukhi@tifr.res.in, rahulnig@theory.tifr.res.in

Abstract: We obtain the complete set of constraints on the moduli of $\mathcal{N} = 4$ superstring compactifications that permit "rare" marginal decays of $\frac{1}{4}$-BPS dyons to take place. The constraints are analysed in some special cases. The analysis extends in a straightforward way to multi-particle decays. We discuss the possible relation between general multi-particle decays and multi-centred black holes.

Keywords: Black Holes in String Theory, D-branes.

Contents

1. Introduction

Recent developments have given us a much better understanding of the degeneracy counting formula for $\frac{1}{4}$-BPS dyons in $\mathcal{N} = 4$ string compactifications.[1] This formula has been considerably refined from its original form in ref. [1] where it was first proposed. One such refinement consists of specifying the integration contour in the degeneracy formula and noting that different contours can lead to different answers for the degeneracy [2, 3] (for a review, see ref. [4]). The effect of varying the integration contours is in the form of discontinuous jumps in the degeneracy whenever the contour crosses a pole in the integration variable and picks up the corresponding residue. This has been interpreted as due to the decay of some $\frac{1}{4}$-BPS dyons into a pair of $\frac{1}{2}$-BPS dyons at curves of marginal stability, which are computed using the BPS mass formula.

Because for large charges the decaying states are black holes, a mechanism is needed to explain exactly how these decay on curves of marginal stability. The answer turns out to be [5, 6] that $\frac{1}{4}$-BPS black holes (for a given set of charges) exist both in single-centre and multi-centre varieties. For the latter, the separations of the centres are determined by the moduli [7]. If we specialise to two-centred dyons with both centres being $\frac{1}{2}$-BPS, then it was shown in ref. [5] that as we approach a curve of marginal stability the two centres fly apart to infinity. On the other side of the curve the constraint equation has no

[1]This formula really computes a supersymmetric index, and in what follows when we say "degeneracy" we will always be referring to this index

solutions. This explains the phenomenon of marginal stability and jumping in the counting formula, in terms of the disintegration of two-centred black holes. It should be noted that the degeneracy of single-centred black holes with the same charges does not vary across moduli space, they exist either everywhere or nowhere.

In these developments, the only type of marginal decay that plays a role is into two $\frac{1}{2}$-BPS final states. Also, the only multi-centred black holes needed to complete the explanation are those with a pair of $\frac{1}{2}$-BPS centres. In ref. [5] the correspondence between these two situations was derived for some special cases, while in ref. [6] it is shown to hold in generality, namely for any charge vectors and any point in the entire $\frac{\mathrm{SL}(2)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(6,22)}{\mathrm{SO}(6)\times\mathrm{SO}(22)}$ moduli space of $\mathcal{N}=4$ compactifications.

However, there are many more types of marginal decays in the theory, and in one sense they are far more generic. These decays are into a pair of $\frac{1}{4}$-BPS final states, or into three or more final states each of which can be $\frac{1}{2}$-BPS or $\frac{1}{4}$-BPS. In another sense these decays are "rare", as it has been shown [8 − 10] (at least for unit-torsion initial dyons) that they take place on curves of marginal stability that have a co-dimension > 1 in the moduli space.[2] Therefore these have been labelled "rare decays". In particular they cannot lead to jumps in the degeneracy (or rather, index) formula.[3] Nevertheless the existence of such decay modes is of importance in understanding the behaviour of dyons as we move around in moduli space, and we will study them here for their own sake as well as for possible interesting physical consequences that they may turn out to have.

In ref. [9] these curves were precisely characterised as circles in the upper half-plane labelled by the parameter τ corresponding to the SL(2)/U(1) factor of the moduli space.[4] These circles depend on the other moduli as well. However, as was demonstrated in refs. [8 − 10], there are additional conditions that need to be imposed on the remaining moduli in order to make the decay possible. These latter conditions have not yet been worked out. In this paper we will obtain these conditions and thereby completely characterise the codimension > 1 subspace on which rare decays can take place.

It is also known that there exist multi-centred dyonic black holes with two $\frac{1}{4}$-BPS centres, or three or more centres each of which can be $\frac{1}{2}$- or $\frac{1}{4}$-BPS. However, because the degeneracy formula does not jump at curves of marginal stability, these multi-centred dyons have not played a role in studies of dyons in $\mathcal{N}=4$ compactifications. In particular they have not been related to marginal decays into two $\frac{1}{4}$-BPS final states or multiple final states, and in fact such a relation does not seem necessary for the state-counting problem. Nevertheless, in what follows we will argue that the relation between curves of marginal stability and multi-centred black holes flying apart is quite generic.

In what follows, we start by briefly reviewing what is known about "rare" marginal

[2] Therefore they should not technically be called "curves", but we use this terminology anyway and hope it does not cause confusion.

[3] For higher-torsion initial dyons the curves can be of codimension 1, but the index is still not expected to jump, because of fermion zero modes. We will focus largely on unit-torsion dyons in this paper.

[4] In the type IIB on $K3\times T^2$ description this τ is the modular parameter of the geometrical torus, hence we sometimes refer to the τ UHP as the "torus moduli space" — although technically it would be more accurate to call it the Teichmüller space of the torus.

decays in $\mathcal{N}=4$ compactifications. Then we find a precise form for the constraints on moduli space in order for such rare decays to take place. We examine and solve these constraints in a variety of special cases, to give a flavour of what they look like. Then using some known results on T-duality orbits, we will obtain the constraints in the general case. Next we recursively identify the loci of marginal stability for multi-particle decays. Finally we examine the special-geometry formula for generic multi-centred black holes and write it in a form that relates their separations to curves of marginal stability for $n \geq 2$-body decays.

2. Marginal stability for $\mathcal{N}=4$ dyons

The electric and magnetic charge vectors of a dyon in an $\mathcal{N}=4$ string compactification are elements of a 28 dimensional integral charge lattice of signature $(6,22)$. The formulae for BPS mass involve a 28×28 matrix L, which in our basis will be taken to be:

$$\begin{pmatrix} 0 & \mathbb{I}_6 & 0 \\ \mathbb{I}_6 & 0 & 0 \\ 0 & 0 & -\mathbb{I}_{16} \end{pmatrix} \tag{2.1}$$

as well as a 28×28 matrix M of moduli satisfying $MLM^T = L$. The inner product of charge vectors appearing in the BPS mass is taken with the matrix $L+M$. In the heterotic basis where the compactification is specified by a constant metric G_{ij}, an antisymmetric tensor field B_{ij} and constant gauge potentials A^I_i (where $i=1,2,\cdots,6$ and $I=1,2,\cdots,16$), this matrix is [11, 12]:

$$L+M = \begin{pmatrix} G^{-1} & 1+G^{-1}(B+C) & G^{-1}A \\ 1+(-B+C)G^{-1} & (G-B+C)G^{-1}(G+B+C) & (G-B+C)G^{-1}A \\ A^TG^{-1} & A^TG^{-1}(G+B+C) & A^TG^{-1}A \end{pmatrix} \tag{2.2}$$

Here C is a symmetric 6×6 matrix constructed from A as $C=\frac{1}{2}A^TA$, more concretely $C_{ij} = \frac{1}{2}A^I_iA^I_j$.

In this basis we parametrise the charge vectors explicitly as:

$$\vec{Q} = \begin{pmatrix} \vec{Q}'_{(6)} \\ \vec{Q}''_{(6)} \\ \vec{Q}'''_{(16)} \end{pmatrix}, \qquad \vec{P} = \begin{pmatrix} \vec{P}'_{(6)} \\ \vec{P}''_{(6)} \\ \vec{P}'''_{(16)} \end{pmatrix} \tag{2.3}$$

where we have broken up the original vectors into three parts with 6,6 and 16 components respectively. In subsequent discussions we will not explicitly write out the subscripts $(6),(16)$ that appear in the above formula.

The BPS mass formula for $\frac{1}{4}$-BPS dyons in $\mathcal{N}=4$ compactifications is as follows [13, 14]:

$$m_{\mathrm{BPS}}(\vec{Q},\vec{P})^2 = \frac{1}{\sqrt{\tau}_2}(\vec{Q}-\bar{\tau}\vec{P})\circ(\vec{Q}-\tau\vec{P}) + 2\sqrt{\tau_2}\sqrt{\Delta(\vec{Q},\vec{P})} \tag{2.4}$$

where

$$\Delta(\vec{Q},\vec{P}) \equiv (Q\circ Q)\ (P\circ P) - (P\circ Q)^2 \tag{2.5}$$

The inner products of charge vectors appearing in this formula are of the form:

$$Q\circ P \equiv \vec{Q}^T(L+M)\vec{P} \tag{2.6}$$

The matrix $L+M$ has 22 zero eigenvalues and therefore the inner product only contains a projected set of 6 components from the original 28 components of the charge vector. Explicitly, the zero eigenvectors take the form:

$$\begin{pmatrix} G+B+C & A^I \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.7}$$

where each column of the above matrix describes an independent zero eigenvector.

It is convenient to replace the inner product on charge vectors in eq. (2.6) by an ordinary product acting on some projected vectors. To do this, define $\sqrt{L+M}$ as a 28×28 matrix satisfying $\sqrt{L+M}^T\sqrt{L+M} = L+M$. This will be ambiguous upto a "gauge" freedom but we will select a specific solution that is particularly useful, namely:

$$\sqrt{L+M} = \begin{pmatrix} E^{-1} & E^{-1}(G+B+C) & E^{-1}A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.8}$$

where E stands for the vielbein: $E_i^a E_j^a = G_{ij}$.

With this matrix it is evident that the projected charges only have their first 6 components nonzero, namely for any arbitrary vectors $\vec{Q},\vec{P}$ the projected vectors $\vec{Q}_R,\vec{P}_R$ defined by:

$$\vec{Q}_R = \sqrt{L+M}\,\vec{Q}, \qquad \vec{P}_R = \sqrt{L+M}\,\vec{P} \tag{2.9}$$

are 6-component vectors. The components of these vectors are moduli dependent and not quantised. On the projected vectors, one only needs to consider ordinary inner products, for example $\vec{Q}_R^T\vec{Q}_R$ is equal (by construction) to $\vec{Q}^T(L+M)\vec{Q}$. Hence in what follows we will denote this quantity either by $\vec{Q}\circ\vec{Q}$ or equivalently by $\vec{Q}_R\cdot\vec{Q}_R$, and analogously for other inner products.

Within the 6-dimensional projected charge space, the electric and magnetic charge vectors of the initial dyon span a 2-dimensional plane. Decay of a $\frac{1}{4}$-BPS dyon into a set of decay products with quantised charge vectors $(\vec{Q}^{(1)},\vec{P}^{(1)}),\cdots,(\vec{Q}^{(n)},\vec{P}^{(n)})$ can take place only when the plane spanned by the projected charge vectors of each decay product coincides with this plane (this is the condition for all states to be mutually $\frac{1}{4}$-BPS):

$$\begin{pmatrix} \vec{Q}_R^{(i)} \\ \vec{P}_R^{(i)} \end{pmatrix} = \begin{pmatrix} m_i & r_i \\ s_i & n_i \end{pmatrix} \begin{pmatrix} \vec{Q}_R \\ \vec{P}_R \end{pmatrix} \tag{2.10}$$

When there are just two decay products and both are $\frac{1}{2}$-BPS, the pair of decay products defines a 2-plane. Charge conservation then implies that this plane coincides with the plane of the original charge vectors, so in this very special case the above requirement imposes no conditions on the moduli. Indeed, the numbers m_i, r_i, s_i, n_i are then integers and the above relation holds between the full (quantised) charge vectors, not only the projected ones. Marginal decay takes place on a wall of marginal stability whose equation is explicitly known (see ref. [2] and references therein). In all other cases, the numbers m_i, r_i, s_i, n_i are non-integral and moduli-dependent. In these cases the above condition puts additional constraints on the background moduli M. Our goal here is to identify these constraints explicitly.

For a two-body decay into $\frac{1}{4}$-BPS constituents, once the constraints are satisfied and we find the numbers m_1, r_1, s_1, n_1 (the corresponding numbers m_2, r_2, s_2, n_2 are determined by charge conservation) the condition for marginal decay is expressed in terms of the curve [9]:

$$\left(\tau_1 - \frac{m_1 - n_1}{2s_1}\right)^2 + \left(\tau_2 + \frac{\mathcal{E}}{2s_1}\right)^2 = \frac{1}{4s_1^2}\Big((m_1 - n_1)^2 + 4r_1 s_1 + \mathcal{E}^2\Big) \tag{2.11}$$

Here we have restricted to the case of unit-torsion dyons, so we have put $m = n = 1$ with respect to the notation in ref. [9]. Also, $\mathcal{E}$ is defined by:

$$\mathcal{E} \equiv \frac{1}{\sqrt{\Delta}}\left(\vec{Q}^{(1)} \circ \vec{P} - \vec{P}^{(1)} \circ \vec{Q}\right) \tag{2.12}$$

Interestingly the numerator of this quantity is the Saha angular momentum between one of the final-state dyons and the initial state, evaluated with respect to the moduli at infinity. Exchanging the role of the two final-state dyons sends $\mathcal{E} \to -\mathcal{E}$. It also sends $m_1 - n_1 \to (1 - m_1) - (1 - n_1) = -(m_1 - n_1)$ and $r_1, s_1 \to -r_1, -s_1$. The curve of marginal stability is invariant under this set of transformations, as it should be.

We now turn to the detailed study general two-body decays into $\frac{1}{4}$-BPS constituents. We will find explicit expressions for the numbers m_1, r_1, s_1, n_1 in terms of the quantised charge vectors $\vec{Q}, \vec{P}, \vec{Q}_1, \vec{P}_1$ and the moduli M. We will also explicitly characterise the loci in moduli space where such rare decays are allowed.

3. Rare dyon decays

3.1 Analysis and implicit solution

It will be useful to define a quartic scalar invariant of four different vectors by:

$$\Delta(\vec{A}, \vec{B}; \vec{C}, \vec{D}) \equiv \det\begin{pmatrix} \vec{A}\circ\vec{C} & \vec{A}\circ\vec{D} \\ \vec{B}\circ\vec{C} & \vec{B}\circ\vec{D}\end{pmatrix} = (\vec{A}\circ\vec{C})(\vec{B}\circ\vec{D}) - (\vec{A}\circ\vec{D})(\vec{B}\circ\vec{C}) \tag{3.1}$$

As explained above, the "$\circ$" product is the moduli-dependent inner product involving the matrix $L + M$. The above quantity is antisymmetric under exchange of the first pair or last pair of vectors, and symmetric under exchange of the two pairs. The quartic invariant of two variables defined earlier is a special case of this new invariant:

$$\Delta(\vec{Q}, \vec{P}) = \Delta(\vec{Q}, \vec{P}; \vec{Q}, \vec{P}) \tag{3.2}$$

Now start with the following vector equation that is part of eq. (2.10):

$$\vec{Q}_R^{(1)} = m_1 \vec{Q}_R + r_1 \vec{P}_R \tag{3.3}$$

Contracting this successively with $\vec{Q}_R$ and $\vec{P}_R$ we find:

$$\begin{aligned} \vec{Q}_R^{(1)} \cdot \vec{Q}_R &= m_1 \vec{Q}_R^2 + r_1 \vec{Q}_R \cdot \vec{P}_R \\ \vec{Q}_R^{(1)} \cdot \vec{P}_R &= m_1 \vec{Q}_R \cdot \vec{P}_R + r_1 \vec{P}_R^2 \end{aligned} \tag{3.4}$$

Multiplying the first equation by $\vec{P}_R^2$ and the second by $\vec{Q}_R \cdot \vec{P}_R$ and subtracting, we find:

$$m_1 \Delta(\vec{Q}_R, \vec{P}_R) = \Delta(\vec{Q}_R, \vec{P}_R; \vec{Q}_R^{(1)}, \vec{P}_R) \tag{3.5}$$

which enables us to solve for m_1. Repeating this process we can solve for r_1, s_1, n_1 leading to the result:

$$\begin{pmatrix} m_1 & r_1 \\ s_1 & n_1 \end{pmatrix} = \frac{1}{\Delta(\vec{Q}_R, \vec{P}_R)} \begin{pmatrix} \Delta(\vec{Q}_R, \vec{P}_R; \vec{Q}_R^{(1)}, \vec{P}_R) & \Delta(\vec{Q}_R, \vec{P}_R; \vec{Q}_R, \vec{Q}_R^{(1)}) \\ \Delta(\vec{Q}_R, \vec{P}_R; \vec{P}_R^{(1)}, \vec{P}_R) & \Delta(\vec{Q}_R, \vec{P}_R; \vec{Q}_R, \vec{P}_R^{(1)}) \end{pmatrix} \tag{3.6}$$

It follows that eq. (2.10) can be expressed as:

$$\begin{pmatrix} \vec{Q}_R^{(1)} \\ \vec{P}_R^{(1)} \end{pmatrix} = \frac{1}{\Delta(\vec{Q}_R, \vec{P}_R)} \begin{pmatrix} \Delta(\vec{Q}_R, \vec{P}_R; \vec{Q}_R^{(1)}, \vec{P}_R) & \Delta(\vec{Q}_R, \vec{P}_R; \vec{Q}_R, \vec{Q}_R^{(1)}) \\ \Delta(\vec{Q}_R, \vec{P}_R; \vec{P}_R^{(1)}, \vec{P}_R) & \Delta(\vec{Q}_R, \vec{P}_R; \vec{Q}_R, \vec{P}_R^{(1)}) \end{pmatrix} \begin{pmatrix} \vec{Q}_R \\ \vec{P}_R \end{pmatrix} \tag{3.7}$$

For fixed, quantised charge vectors $\vec{Q}, \vec{P}$ of the initial dyon and $\vec{Q}^{(1)}, \vec{P}^{(1)}$ of the first decay product (the charge of the second product is determined by charge conservation), the above equation provides a set of constraints on the moduli that must be satisfied for the $\frac{1}{4} \to \frac{1}{4} + \frac{1}{4}$ decay to be possible.

In the above form, it is rather difficult to disentangle the constraints or to physically understand their significance. Therefore we will consider a number of special cases. Along the way we will see the advantage of using T-duality to bring the charges into a convenient form and performing the analysis in that basis. Finally we write down the explicit constraint equation in the general case, again in the chosen T-duality basis.

3.2 Explicit solution: special cases

(i) $\frac{1}{2}$-BPS final states. The case where the decay products are $\frac{1}{2}$-BPS should provide no constraints on the moduli as this is a "non-rare" decay. This provides a check on our equations. Inserting the $\frac{1}{2}$-BPS conditions:

$$\vec{P}^{(1)} = \frac{k_1}{l_1} \vec{Q}^{(1)}, \quad \vec{P}^{(2)} = \frac{k_2}{l_2} \vec{Q}^{(2)} \tag{3.8}$$

with k_i, l_i integers, we find that:

$$\begin{pmatrix} m_1 & r_1 \\ s_1 & n_1 \end{pmatrix} = \left(\frac{k_2}{l_2} - \frac{k_1}{l_1} \right) \frac{\Delta(\vec{Q}_R^{(1)}, \vec{Q}_R^{(2)})}{\Delta(\vec{Q}_R, \vec{P}_R)} \begin{pmatrix} \frac{k_2}{l_2} & -1 \\ \frac{k_1 k_2}{l_1 l_2} & -\frac{k_1}{l_1} \end{pmatrix} \tag{3.9}$$

We also have:

$$\Delta(\vec{Q}_R,\vec{P}_R) = \left(\frac{k_2}{l_2}-\frac{k_1}{l_1}\right)^2 \Delta(\vec{Q}_R^{(1)},\vec{Q}_R^{(2)}) \tag{3.10}$$

Substituting in the above equation, we find:

$$\begin{pmatrix} m_1 & r_1 \\ s_1 & n_1 \end{pmatrix} = \frac{1}{k_2 l_1 - k_1 l_2}\begin{pmatrix} k_2 l_1 & -l_1 l_2 \\ k_1 k_2 & -k_1 l_2 \end{pmatrix} \tag{3.11}$$

At this stage all moduli-dependence has disappeared from the matrix, and equation eq. (2.10) indeed provides no constraints on the moduli. Rather, it reduces to an identity. It is also easy to see that $k_1 l_2 - k_2 l_1$ divides the torsion of the original dyon, so if we are also considering the unit-torsion case then $k_1 l_2 - k_2 l_1 = 1$ and m_1, r_1, s_1, n_1 are all manifestly integral [2].

(ii) Special charges and moduli. The next special case we will study has a restricted set of charges. Additionally, some of the background moduli are set to a specific value, namely zero in the chosen coordinates. We then examine the constraints on the remaining moduli. In choosing special values for the moduli, we should in principle avoid loci of enhanced gauge symmetry where the dyons we are studying would become massless.

Let us restrict ourselves to special initial-state charges given by:

$$\vec{Q}' = (Q'_1, 0, \cdots, 0), \qquad \vec{Q}'' = (Q''_1, 0, \cdots, 0), \qquad \vec{Q}''' = 0 \tag{3.12}$$

and

$$\vec{P}' = (0, P'_2, 0, \cdots, 0), \qquad \vec{P}'' = (0, P''_2, 0, \cdots, 0), \qquad \vec{P}''' = 0 \tag{3.13}$$

Next we set $B_{ij} = 0 = A_i^I$ as well as $G_{ij} = 0,\ i \neq j$. The above restrictions allow us to choose the orthonormal frames E_{ai} to be diagonal:

$$E_{ii} = R_i,\ i = 1, 2, \cdots, 6 \tag{3.14}$$

with R_i the radii of the six compactified directions in the heterotic basis.

In the restricted subspace of moduli space that we are considering here, the matrix $\sqrt{L+M}$ reduces to:

$$\sqrt{L+M} = \begin{pmatrix} E^{-1} & E & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.15}$$

with E given as in eq. (3.14). Therefore the projected initial-state charge vectors are:

$$\vec{Q}_R = \begin{pmatrix} \frac{Q'_1}{R_1} + Q''_1 R_1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}, \qquad \vec{P}_R = \begin{pmatrix} 0 \\ \frac{P'_2}{R_2} + P''_2 R_2 \\ 0 \\ \cdots \\ 0 \end{pmatrix} \tag{3.16}$$

For this configuration we clearly have $\vec{Q}_R \cdot \vec{P}_R = 0$ and therefore the quartic invariant Δ is:

$$\Delta(Q_R, P_R) = \left(\frac{Q_1'}{R_1} + Q_1'' R_1\right)^2 \left(\frac{P_2'}{R_2} + P_2'' R_2\right)^2 \tag{3.17}$$

We take the decay products to have generic charges $\vec{Q}^{(1)}, \vec{P}^{(1)}$ and $\vec{Q}^{(2)}, \vec{P}^{(2)}$ subject of course to the requirement that they add up to $\vec{Q}, \vec{P}$. We then have:

$$\vec{Q}_R^{(1)} = \begin{pmatrix} \frac{Q_1^{(1)\prime}}{R_1} + Q_1^{(1)\prime\prime} R_1 \\ \frac{Q_2^{(1)\prime}}{R_2} + Q_2^{(1)\prime\prime} R_2 \\ \cdots \\ \frac{Q_6^{(1)\prime}}{R_6} + Q_6^{(1)\prime\prime} R_6 \end{pmatrix}, \quad \vec{P}_R^{(1)} = \begin{pmatrix} \frac{P_1^{(1)\prime}}{R_1} + P_1^{(1)\prime\prime} R_1 \\ \frac{P_2^{(1)\prime}}{R_2} + P_2^{(1)\prime\prime} R_2 \\ \cdots \\ \frac{P_6^{(1)\prime}}{R_6} + P_6^{(1)\prime\prime} R_6 \end{pmatrix} \tag{3.18}$$

Now we can compute the quartic invariants appearing in eq. (3.7):

$$\begin{aligned}
\Delta(\vec{Q}_R, \vec{P}_R; \vec{Q}_R^{(1)}, \vec{P}_R) &= \left(\frac{Q_1'}{R_1} + Q_1'' R_1\right) \left(\frac{Q_1^{(1)\prime}}{R_1} + Q_1^{(1)\prime\prime} R_1\right) \left(\frac{P_2'}{R_2} + P_2'' R_2\right)^2 \\
\Delta(\vec{Q}_R, \vec{P}_R; \vec{Q}_R, \vec{Q}_R^{(1)}) &= \left(\frac{Q_1'}{R_1} + Q_1'' R_1\right)^2 \left(\frac{Q_2^{(1)\prime}}{R_2} + Q_2^{(1)\prime\prime} R_2\right) \left(\frac{P_2'}{R_2} + P_2'' R_2\right) \\
\Delta(\vec{Q}_R, \vec{P}_R; \vec{P}_R^{(1)}, \vec{P}_R) &= \left(\frac{Q_1'}{R_1} + Q_1'' R_1\right) \left(\frac{P_1^{(1)\prime}}{R_1} + P_1^{(1)\prime\prime} R_1\right) \left(\frac{P_2'}{R_2} + P_2'' R_2\right)^2 \\
\Delta(\vec{Q}_R, \vec{P}_R; \vec{Q}_R, \vec{P}_R^{(1)}) &= \left(\frac{Q_1'}{R_1} + Q_1'' R_1\right)^2 \left(\frac{P_2^{(1)\prime}}{R_2} + P_2^{(1)\prime\prime} R_2\right) \left(\frac{P_2'}{R_2} + P_2'' R_2\right)
\end{aligned}$$

Had we not taken E to be diagonal, the expressions above would have quickly become very complicated to write down.

Inserting the above expressions, and cancelling some common factors, the constraint equation eq. (3.7) becomes:

$$\begin{aligned}
\left(\frac{Q_1'}{R_1} + Q_1'' R_1\right) \left(\frac{P_2'}{R_2} + P_2'' R_2\right) \vec{Q}_R^{(1)} &= \left(\frac{Q_1^{(1)\prime}}{R_1} + Q_1^{(1)\prime\prime} R_1\right) \left(\frac{P_2'}{R_2} + P_2'' R_2\right) \vec{Q}_R \\
&\quad + \left(\frac{Q_1'}{R_1} + Q_1'' R_1\right) \left(\frac{Q_2^{(1)\prime}}{R_2} + Q_2^{(1)\prime\prime} R_2\right) \vec{P}_R \\
\left(\frac{Q_1'}{R_1} + Q_1'' R_1\right) \left(\frac{P_2'}{R_2} + P_2'' R_2\right) \vec{P}_R^{(1)} &= \left(\frac{P_1^{(1)\prime}}{R_1} + P_1^{(1)\prime\prime} R_1\right) \left(\frac{P_2'}{R_2} + P_2'' R_2\right) \vec{Q}_R \\
&\quad + \left(\frac{Q_1'}{R_1} + Q_1'' R_1\right) \left(\frac{P_2^{(1)\prime}}{R_2} + P_2^{(1)\prime\prime} R_2\right) \vec{P}_R
\end{aligned} \tag{3.19}$$

These are 6+6 equations. However, the first two components of each set are identically satisfied, as one can easily check. This is expected, and follows from the structure of eq. (2.10) from which m_1, r_1, s_1, n_1 were determined. The remaining four components of

JHEP12(2008)056

each equation give the desired constraints on the moduli. Because of the way we have chosen $\vec{Q}, \vec{P}$, the r.h.s. already vanishes on components 3 to 6, so the constraint is simply that the l.h.s. vanishes. That in turn sets to zero the components 3 to 6 of the vectors $\vec{Q}_R^{(1)}$ and $\vec{P}_R^{(1)}$. Thus we find the constraints:

$$\begin{aligned} \frac{Q_i^{(1)'}}{R_i} + Q_i^{(1)''} R_i = 0, \quad i = 3,4,5,6 \\ \frac{P_i^{(1)'}}{R_i} + P_i^{(1)''} R_i = 0, \quad i = 3,4,5,6 \end{aligned} \tag{3.20}$$

If the components of $\vec{Q}^{(1)}, \vec{P}^{(1)}$ are all nonvanishing, this implies that:

$$R_i = \sqrt{-\frac{Q_i^{(1)'}}{Q_i^{(1)''}}} = \sqrt{-\frac{P_i^{(1)'}}{P_i^{(1)''}}}, \quad i = 3,4,5,6 \tag{3.21}$$

In this special case the constraint equations have some particular features. First of all, for generic charge vectors $\vec{Q}^{(1)}$ and $\vec{P}^{(1)}$, there are no solutions. To have any solutions at all, one must choose the charges of the decay products in such a way that the second equality in the above equation can be satisfied. In other words, the sign of $Q_i^{(1)'}$ and $Q_i^{(1)''}$ must be opposite (for $i = 3,4,5,6$), and the same has to be true for $P^{(1)}$. In this case we find four constraints on the moduli, which fix the compactification radii R_3, R_4, R_5, R_6.

For this special case, the numbers m_1, r_1, s_1, n_1 in eq. (2.10) are given by:

$$\begin{aligned} m_1 &= \frac{\frac{Q_1^{(1)'}}{R_1} + Q_1^{(1)''} R_1}{\frac{Q_1'}{R_1} + Q_1'' R_1}, \qquad r_1 = \frac{\frac{Q_2^{(1)'}}{R_2} + Q_2^{(1)''} R_2}{\frac{P_2'}{R_2} + P_2'' R_2} \\ s_1 &= \frac{\frac{P_1^{(1)'}}{R_1} + P_1^{(1)''} R_1}{\frac{Q_1'}{R_1} + Q_1'' R_1}, \qquad n_1 = \frac{\frac{P_2^{(1)'}}{R_2} + P_2^{(1)''} R_2}{\frac{P_2'}{R_2} + P_2'' R_2} \end{aligned} \tag{3.22}$$

We see that m_1, s_1 depend only on R_1 and r_1, n_1 depend only on R_2.

So far the decay products were taken to have generic charges (consistent of course with charge conservation). The situation changes if we choose less generic decay products. Earlier we took all components of $\vec{Q}^{(1)}, \vec{P}^{(1)}$ are nonvanishing. However if $Q_i^{(1)'} = Q_i^{(1)''} = P_i^{(1)'} = P_i^{(1)''} = 0$ for any $i \in 3,4,5,6$ then the corresponding constraint eq. (3.20) is trivially satisfied. In this situation we will have a reduced number of constraints. As an example if the above situation holds for all directions except $i = 3$ and if $\frac{Q_3^{(1)'}}{Q_3^{(1)''}} = \frac{P_3^{(1)'}}{P_3^{(1)''}}$ then there is only a single constraint coming from the above equations. The curve of marginal stability provides one more constraint, so the decay will take place on a codimension-2 subspace of the restricted moduli space in which we are working for this class of examples. The fact that in some situations there are no solutions (for example if we do not satisfy that $Q_i^{(1)'}$ and $Q_i^{(1)''}$ have opposite signs for $i = 3,4,5,6$) simply means that our restricted moduli space fails to intersect the marginal stability locus in that case.

If the charges $Q_i^{(1)'}, Q_i^{(1)''}, P_i^{(1)'}, P_i^{(1)''}$ vanish for all $i \in 3,4,5,6$ then there are no constraints (beyond the curve of marginal stability). This can correspond to two distinct situations. One is that the final states are now both $\frac{1}{2}$-BPS. The other possibility is that they are still $\frac{1}{4}$-BPS, but the apparent contradiction of having no constraints on moduli is resolved by the fact that we are already in a restricted subspace of the moduli space.[5]

(iii) General charges, "diagonal" moduli. In this subsection we study rare decays allowing for completely general charges $\vec{Q}, \vec{P}$, but we will restrict the moduli so that the formulae are tractable. The situation turns out to be rather similar to the case studied in the previous subsection.

Considerable simplification can be brought about in the formulae by using some known results on T-duality orbits from ref. [15] (as reviewed in appendix A of [16]). For this purpose we first change basis from the L matrix used in ref. [16]:

$$L' = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \sigma_1 & 0 & \cdots & 0 \\ 0 & & \cdots & & -L_{E_8} & & 0 \\ 0 & & \cdots & & 0 & & -L_{E_8} \end{pmatrix} \tag{3.23}$$

to the one we have defined in eq. (2.1). Here σ_1 is a Pauli matrix, which occurs 6 times in the above, and L_{E_8} is the Cartan matrix of E_8.

In fact using T-duality we will be able to restrict to charge vectors that have the last 16 components vanishing, therefore we can ignore these components and work in a space of 12-component vectors. We then use a 12×12 matrix X that satisfies

$$XLX^T = L' \tag{3.24}$$

to map the equations in ref. [16] to our basis.

Now the relevant result of T-duality orbits states that any pair of primitive charge vectors $\vec{Q}, \vec{P}$ can be brought via T-duality to the form:

$$\begin{aligned} \vec{Q}' &= (Q_1', 0, \cdots, 0), \qquad & \vec{Q}'' &= (Q_1'', 0, \cdots, 0), \qquad & \vec{Q}''' &= 0 \\ \vec{P}' &= (P_1', P_2', 0, \cdots, 0), \qquad & \vec{P}'' &= (P_1'', P_2'', 0, \cdots, 0), \qquad & \vec{P}''' &= 0 \end{aligned} \tag{3.25}$$

This is close to our previous special case, but with P_1', P_1'' turned on. It is no longer a special case but represents the general case in a special basis.

As in the previous example, we restrict the moduli by requiring $A_i^I = B_{ij} = 0$ and $G_{ij} = 0,\ i \neq j$. Then one finds the projected charges to be:

$$\vec{Q}_R = \begin{pmatrix} \frac{Q_1'}{R_1} + Q_1'' R_1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}, \qquad \vec{P}_R = \begin{pmatrix} \frac{P_1'}{R_1} + P_1'' R_1 \\ \frac{P_2'}{R_2} + P_2'' R_2 \\ 0 \\ \cdots \\ 0 \end{pmatrix} \tag{3.26}$$

[5] We thank the referee for pointing out an erroneous statement in a previous version.

The quartic invariant is then found to be:

$$\Delta(Q_R,P_R)=\left(\frac{Q_1'}{R_1}+Q_1''R_1\right)^2\left(\frac{P_2'}{R_2}+P_2''R_2\right)^2 \tag{3.27}$$

which is actually the *same* as in the previous, simpler case where we chose a special subset of charges. Computing m_1, r_1, s_1, n_1 as in the previous subsection and inserting them back, the constraint equation can now be written:

$$\begin{aligned}
&\left(\frac{Q_1'}{R_1}+Q_1''R_1\right)\left(\frac{P_2'}{R_2}+P_2''R_2\right)\vec{Q}_R^{(1)}= \\
&\quad\left[\left(\frac{Q_1^{(1)'}}{R_1}+Q_1^{(1)''}R_1\right)\left(\frac{P_2'}{R_2}+P_2''R_2\right)-\left(\frac{Q_2^{(1)'}}{R_2}+Q_2^{(1)''}R_2\right)\left(\frac{P_1'}{R_1}+P_1''R_1\right)\right]\vec{Q}_R \\
&\qquad\qquad+\left(\frac{Q_1'}{R_1}+Q_1''R_1\right)\left(\frac{Q_2^{(1)'}}{R_2}+Q_2^{(1)''}R_2\right)\vec{P}_R \\
&\left(\frac{Q_1'}{R_1}+Q_1''R_1\right)\left(\frac{P_2'}{R_2}+P_2''R_2\right)\vec{P}_R^{(1)}= \\
&\quad\left[\left(\frac{P_1^{(1)'}}{R_1}+P_1^{(1)''}R_1\right)\left(\frac{P_2'}{R_2}+P_2''R_2\right)-\left(\frac{P_2^{(1)'}}{R_2}+P_2^{(1)''}R_2\right)\left(\frac{P_1'}{R_1}+P_1''R_1\right)\right]\vec{Q}_R \\
&\qquad\qquad+\left(\frac{Q_1'}{R_1}+Q_1''R_1\right)\left(\frac{P_2^{(1)'}}{R_2}+P_2^{(1)''}R_2\right)\vec{P}_R
\end{aligned} \tag{3.28}$$

These equations are slightly more complicated than the previous case for which we had $\vec{Q}\circ\vec{P}=0$, but the extra complication is only in the first two components, which are again trivially satisfied. For the remaining components we find:

$$\begin{aligned}
&\frac{Q_i^{(1)'}}{R_i}+Q_i^{(1)''}R_i=0, \quad i=3,4,5,6 \\
&\frac{P_i^{(1)'}}{R_i}+P_i^{(1)''}R_i=0, \quad i=3,4,5,6
\end{aligned} \tag{3.29}$$

These are exactly the *same* as the constraints we found in the previous case. The analysis is therefore also the same: the constraints cannot be satisfied for generic charges because our restricted moduli space need not intersect the marginal stability locus. When they can be satisfied there are at most four constraints, though there will be less if some of the decay product charges vanish.

3.3 General charges, "triangular" moduli

In this subsection we restrict the moduli in the most minimal way consistent with finding a simple form of the constraint equation. The restriction will be a kind of "triangularity" condition:

$$(G+B+C)_{i1}=(G+B+C)_{i2}=0, \quad i=3,4,5,6 \tag{3.30}$$

with no separate constraints on G, B, A other than the above.

As before, we use T-duality to put the initial charges into the form of eq. (3.25). Thereafter, we are still free to make T-duality transformations involving the last four components of $\vec{Q}'$ and $\vec{Q}''$ and all 16 components of $\vec{Q}'''$. The T-duality group is thus restricted to an $\mathrm{SO}(4,20;\mathbb{Z})$. These transformations will affect the charges of the decay products while leaving the initial dyon unchanged. Using them we bring the electric charges of the first decay product to the form:

$$\begin{aligned}
\vec{Q}^{(1)'} &= (Q_1^{(1)'}, Q_2^{(1)'}, Q_3^{(1)'}, 0, \cdots, 0), \\
\vec{Q}^{(1)''} &= (Q_1^{(1)''}, Q_2^{(1)''}, Q_3^{(1)''}, 0, \cdots, 0), \\
\vec{Q}^{(1)'''} &= 0
\end{aligned} \tag{3.31}$$

Finally we use an $\mathrm{SO}(3,19;\mathbb{Z})$ subgroup of T-duality that preserves all the charge vectors that we have so far fixed, to bring the magnetic charges of the first decay product to the form:

$$\begin{aligned}
\vec{P}^{(1)'} &= (P_1^{(1)'}, P_2^{(1)'}, P_3^{(1)'}, P_4^{(1)'}, 0, \cdots, 0), \\
\vec{P}^{(1)''} &= (P_1^{(1)''}, P_2^{(1)''}, P_3^{(1)''}, P_4^{(1)''}, 0, \cdots, 0), \\
\vec{P}^{(1)'''} &= 0
\end{aligned} \tag{3.32}$$

The charges of the second decay product are determined by charge conservation.

Now we use the form of the projection matrix $\sqrt{L+M}$ and write out eq. (2.10) explicitly, after first multiplying through by E_{ij}:

$$\begin{aligned}
Q_i^{(1)'} + (G+B+C)_{ij} Q_j^{(1)''} &= m_1 Q_i' + m_1 (G+B+C)_{ij} Q_j'' \\
&\quad + r_1 P_i' + r_1 (G+B+C)_{ij} P_k'' \\
P_i^{(1)'} + (G+B+C)_{ij} P_j^{(1)''} &= s_1 Q_i' + s_1 (G+B+C)_{ij} Q_j'' \\
&\quad + n_1 P_i' + n_1 (G+B+C)_{ij} P_j''
\end{aligned} \tag{3.33}$$

This is a set of $6+6$ equations. Recall that $C_{ij} = A_i^I A_j^I$.

We immediately see that for our choice of T-duality frame for the initial charges, as well as using the "triangularity" condition, the r.h.s. of the above equations vanishes for $i = 3,4,5,6$. Hence we find the constraint equations still in a relatively simple form:

$$\begin{aligned}
Q_i^{(1)'} + (G+B+C)_{ij} Q_j^{(1)''} &= 0, \quad i = 3,4,5,6 \\
P_i^{(1)'} + (G+B+C)_{ij} P_j^{(1)''} &= 0, \quad i = 3,4,5,6
\end{aligned} \tag{3.34}$$

These are then the 4+4 constraints on rare dyon decays, though still with the triangularity restriction on moduli and in a specific T-duality frame. They must be supplemented by the curve of marginal stability, for which we need to know the numbers m_1, r_1, s_1, n_1.

The first two components of each line of equations eq. (3.33) determine the values of m_1, r_1, s_1, n_1. From the first line of those equations we find:

$$\begin{aligned}
Q_1^{(1)'} + (G+B+C)_{1i} \vec{Q}_i^{(1)''} &= m_1 Q_1' + m_1 (G+B+C)_{1i} Q_i'' + r_1 P_1' + r_1 (G+B+C)_{1i} P_i'' \\
Q_2^{(1)'} + (G+B+C)_{2i} Q_i^{(1)''} &= r_1 P_2' + r_1 (G+B+C)_{2i} P_i''
\end{aligned} \tag{3.35}$$

Solving for r_1 from the second equation above, we get:

$$r_1 = \frac{Q_2^{(1)'} + (G+B+C)_{2i} Q_i^{(1)''}}{P_2' + (G+B+C)_{2i} P_i''} \tag{3.36}$$

Inserting this in the first equation determines m_1:

$$\begin{aligned} m_1 = & \left(P_2' + (G+B+C)_{2i} P_i''\right)^{-1} \left(Q_1' + (G+B+C)_{1i} Q_i''\right)^{-1} \times \\ & \times \Big[\left(Q_1^{(1)'} + (G+B+C)_{1i} Q_i^{(1)''}\right) \left(P_2' + (G+B+C)_{2i} P_i''\right) \\ & - \left(Q_2^{(1)'} + (G+B+C)_{2i} Q_i^{(1)''}\right) \left(P_1' + (G+B+C)_{1i} P_i''\right) \Big] \end{aligned} \tag{3.37}$$

Similarly we solve for s_1, n_1 from the second line of eq. (3.33) and find:

$$\begin{aligned} n_1 = & \frac{P_2^{(1)'} + (G+B+C)_{2i} P_i^{(1)''}}{P_2' + (G+B+C)_{2i} P_i''} \\ s_1 = & \left(P_2' + (G+B+C)_{2i} P_i''\right)^{-1} \left(Q_1' + (G+B+C)_{1i} Q_i''\right)^{-1} \times \\ & \times \Big[\left(P_1^{(1)'} + (G+B+C)_{1i} P_i^{(1)''}\right) \left(P_2' + (G+B+C)_{2i} P_i''\right) \\ & - \left(P_2^{(1)'} + (G+B+C)_{2i} P_i^{(1)''}\right) \left(P_1' + r(G+B+C)_{1i} P_i\right) \Big] \end{aligned} \tag{3.38}$$

Admittedly these are somewhat complicated expressions for the numbers m_1, r_1, s_1, n_1 that one needs to plug in to determine the curve of marginal stability on the torus moduli space. It is conceivable that a more opportune choice of variables could simply them further. Nevertheless, the constraints eq. (3.34) on the remaining moduli are rather simple.

3.4 Explicit solution: the general case

We now turn to the case where the initial and final charges are completely general and the moduli are generic as well. Most of the relevant analysis has already been done in previous subsections and it only remains to write down the result. However, as we will see, the equations rapidly become messy – despite the use of T-duality transformations - once we use completely general moduli.

Let us again start by writing out eq. (2.10) explicitly, but now without any condition on the moduli. After multiplying through by E_{ij}, we find the equations:

$$\begin{aligned} Q_i^{(1)'} + (G+B+C)_{ij} Q_j^{(1)''} &= m_1 Q_i' + m_1 (G+B+C)_{ij} Q_j'' \\ &\quad + r_1 P_i' + r(G+B+C)_{ij} P_k'' \\ P_i^{(1)'} + (G+B+C)_{ij} P_j^{(1)''} &= s_1 Q_i' + s_1 (G+B+C)_{ij} Q_j'' \\ &\quad + n_1 P_i' + n_1 (G+B+C)_{ij} P_j'' \end{aligned} \tag{3.39}$$

which are actually the same as eq. (3.33) that we had before. The difference is that the r.h.s. no longer vanishes for any of the components (earlier that was guaranteed by the triangularity condition that we had assumed on the moduli). Notice that even in the most general case, we have gained something by fixing the initial and final state charges using T-duality. The last 16 components of these charges have all been set to 0, and the result is that most of the terms involving the gauge field moduli A_i^I have disappeared. The only appearance of these moduli is through $C_{ij} = A_i^I A_j^I$ which in turn only appears in the combination $G + B + C$.

This time our strategy will be to choose any 4 equations from the above set of 12 to determine the variables m_1, n_1, r_1, s_1. Then in the remaining 8 equations we insert these values for the variables and obtain the desired constraint equations. Picking the first 2 components for each charge vector, we find:

$$\begin{aligned} Q_1^{(1)'} + (G+B+C)_{1i} Q_i^{(1)''} &= m_1 Q_1' + m_1 (G+B+C)_{1i} Q_i'' + r_1 P_1' + r(G+B+C)_{1i} P_i'' \\ Q_2^{(1)'} + (G+B+C)_{2i} Q_i^{(1)''} &= r_1 P_2' + r_1 (G+B+C)_{2i} P_i'' \end{aligned} \tag{3.40}$$

Solving for r_1 from the second equation above, we get:

$$r_1 = \frac{Q_2^{(1)'} + (G+B+C)_{2i} Q_i^{(1)''}}{P_2' + (G+B+C)_{2i} P_i''} \tag{3.41}$$

and inserting this in the first equation, we find m_1:

$$\begin{aligned} m_1 = & \Big(P_2' + (G+B+C)_{2i} P_i''\Big)^{-1} \Big(Q_1' + (G+B+C)_{1i} Q_i''\Big)^{-1} \times \\ & \Big[\Big(Q_1^{(1)'} + (G+B+C)_{1i} Q_i^{(1)''}\Big)\Big(P_2' + (G+B+C)_{2i} P_i''\Big) \\ & \quad - \Big(Q_2^{(1)'} + (G+B+C)_{2i} \overrightarrow{Q}_i^{(1)''}\Big)\Big(P_1' + r(G+B+C)_{1i} P_i\Big)\Big] \end{aligned} \tag{3.42}$$

Similarly we solve for s_1, n_1 from the second equation and find:

$$n_1 = \frac{P_2^{(1)'} + (G+B+C)_{2i} P_i^{(1)''}}{P_2' + (G+B+C)_{2i} P_i''} \tag{3.43}$$

and

$$\begin{aligned} s_1 = & \Big(P_2' + (G+B+C)_{2i} P_i''\Big)^{-1} \Big(Q_1' + (G+B+C)_{1i} Q_i''\Big)^{-1} \times \\ & \Big[\Big(P_1^{(1)'} + (G+B+C)_{1i} P_i^{(1)''}\Big)\Big(P_2' + (G+B+C)_{2i} P_i''\Big) \\ & \quad - \Big(P_2^{(1)'} + (G+B+C)_{2i} P_i^{(1)''}\Big)\Big(P_1' + (G+B+C)_{1i} P_i\Big)\Big] \end{aligned} \tag{3.44}$$

We feed in these values of m_1, n_1, r_1, s_1 into the remaining 8 equations to find the most general constraint equations on the moduli:

$$\begin{aligned} Q_i^{(1)'} + (G+B+C)_{ij} Q_j^{(1)''} &= m_1 \Big(Q_i' + (G+B+C)_{ij} Q_j''\Big) + r_1 \Big(P_i' + (G+B+C)_{ij} P_j''\Big) \\ P_i^{(1)'} + (G+B+C)_{ij} P_j^{(1)''} &= s_1 \Big(Q_i' + (G+B+C)_{ij} Q_j''\Big) + n_1 \Big(P_i' + (G+B+C)_{ij} P_j''\Big) \end{aligned} \tag{3.45}$$

here $i = 3, 4, 5, 6$, and m_1, n_1, r_1, s_1 are given in the above equations. We see that the values of m_1, n_1, r_1, s_1 come out the same as in the previous special case, however the constraints are much more complicated and — unlike in all the previous special cases – depend explicitly on these numbers. Nevertheless, the above equations embody the most general kinematic constraints on moduli space to allow a two-body decay of a dyon of charges $\vec{Q}, \vec{P}$ into $\frac{1}{4}$-BPS final state with charges $\vec{Q}^{(1)}$ and $\vec{P}^{(1)}$ (the charges of the second state being, as always, determined by charge conservation). It is quite conceivable that a more detailed study of possible T-duality bases will allow us to further simplify the most general case, and we leave such an investigation for the future.

4. Multi-particle decays

So far in this work, as well as in previous work [9], we have written down conditions for decay of a dyon into two $\frac{1}{4}$-BPS final states. One could certainly imagine extending these considerations to three or more final states. Indeed, it turns out rather simple to do so and we will here discuss an iterative way to obtain the relevant formulae.

Consider the decay of a dyon of charges $(\vec{Q}, \vec{P})$ into n decay products of charges $(\vec{Q}^{(1)}, \vec{P}^{(1)}), (\vec{Q}^{(2)}, \vec{P}^{(2)}), \cdots (\vec{Q}^{(n)}, \vec{P}^{(n)})$. The condition for marginality of such a decay is the condition for the original dyon to go into two decay products of charges $(\vec{Q}^{(1)}, \vec{P}^{(1)})$ and $\sum_{i=2}^{n}(\vec{Q}^{(i)}, \vec{P}^{(i)})$, along with the condition for the second decay product to further decay into say $(\vec{Q}^{(2)}, \vec{P}^{(2)})$ and $\sum_{i=3}^{n}(\vec{Q}^{(i)}, \vec{P}^{(i)})$. The latter condition must in turn be iterated. Each of these is a two-body decay (with both final states being $\frac{1}{4}$-BPS) so we already know the condition for each one to take place. The intersection of all these loci will give the marginal stability locus for the multiparticle decay.

There is a simpler way to iterate the condition. Instead of looking at the curve where the second decay product decays into further subconstituents, as above, we can simply consider the collection of all marginal stability loci for the decays:

$$\begin{pmatrix} \vec{Q}_R \\ \vec{P}_R \end{pmatrix} \to \begin{pmatrix} \vec{Q}_R^{(i)} \\ \vec{P}_R^{(i)} \end{pmatrix} + \begin{pmatrix} \vec{Q}_R - \vec{Q}_R^{(i)} \\ \vec{P}_R - \vec{P}_R^{(i)} \end{pmatrix}, \quad i = 1, 2, \cdots, n \tag{4.1}$$

For each of these, the curve is precisely eq. (2.11) with the subscript "1" replaced by "i". We write it as:

$$\mathcal{C}(m_i, r_i, s_i, n_i) \equiv \left(\tau_1 - \frac{m_i - n_i}{2s_i}\right)^2 + \left(\tau_2 + \frac{\mathcal{E}_i}{2s_i}\right)^2 - \frac{1}{4s_i^2}\Big((m_i - n_i)^2 + 4r_i s_i + \mathcal{E}_i^2\Big) = 0 \tag{4.2}$$

where

$$\mathcal{E}_i \equiv \frac{1}{\sqrt{\Delta}}\left(\vec{Q}^{(i)} \circ \vec{P} - \vec{P}^{(i)} \circ \vec{Q}\right) \tag{4.3}$$

In addition to this curve we have the constraints on the remaining moduli as in section 3 above. Those too can be expressed in terms of the single decay product labelled "i". Now to find the condition for a multi-dyon decay, we simply take the intersection of all these loci of marginal stability. As the number of final states increases, we will generically find loci of marginal stability of increasing codimension.

5. Multi-centred black holes

It was argued in refs. [5, 6] that the curves of marginal stability for decays of the form:

$$\frac{1}{4}\text{-BPS} \to \frac{1}{2}\text{-BPS} + \frac{1}{2}\text{-BPS} \tag{5.1}$$

are also the curves of disintegration for two-centred $\frac{1}{4}$-BPS black holes whose centres are individually $\frac{1}{2}$-BPS. The method used in these works, which we will summarise and extend below, was to use a constraint equation due to Denef [7] to express the separation between the centres of such a black hole in terms of charges and moduli. Requiring that the separation be infinite places a condition on charges and moduli which turns out to be precisely the curve of marginal stability, eq. (2.11), specialised to this decay.

Now Denef's constraint equation is not confined to two-centred black holes alone, but applies to any number of centres. It has a different limitation: it is defined in the context of $\mathcal{N}=2$, rather than $\mathcal{N}=4$ compactifications, and relies on special geometry. Nevertheless, for the cases to which it applies, we can certainly use it in the $\mathcal{N}=4$ context. We will do so and will find that the curves of marginal stability for generic decays to n final states, which we discussed in section 4 above, are precisely reproduced by the constraint equations for multi-centred black holes. This suggests a more generic relationship between multi-particle decays and multi-centred black holes than has been previously considered.

To see this, we need to review the constraint equation of ref. [7] for multi-centred dyons, that was re-discussed in the $\mathcal{N}=4$ context in the two-centred case in ref. [5].[6] Let $p^{(i)I}, q_I^{(i)}$ be the charges of the i-th centre where $i = 1, 2, \cdots, N$. These charges are expressed in the special-geometry basis.[7] Let the 3-vector $\vec{r}_i$ be the location of the i-th centre. And let the moduli be encoded in the standard holomorphic special-geometry variables X^I, F_I. Then the constraint equations are:

$$p^{(i)I}\,\mathrm{Im}\,(F_{I\infty}) - q_I^{(i)}\,\mathrm{Im}\,(X^I_\infty) + \frac{1}{2}\sum_{j\neq i}\frac{p^{(i)I}q_I^{(j)} - q_I^{(i)}p^{(j)I}}{|\vec{r}_i - \vec{r}_j|} = 0 \tag{5.2}$$

Here the subscript ∞ indicates that the corresponding moduli are measured at spatial infinity (for brevity of notation we will drop it when there is no risk of ambiguity). Note that the numerators inside the summation correspond to the Saha angular momentum between each pair of centres.

These are N equations for $\binom{N}{2}$ pairwise distances between the centres. We analyse them following the procedure in ref. [5] for the two-centred case. First of all, one of the equations is redundant. Adding all the equations, we find:

$$p^I\,\mathrm{Im}\,(F_{I\infty}) - q_I\,\mathrm{Im}\,(X^I_\infty) = 0 \tag{5.3}$$

[6] A sign in equation (3.2) of ref. [5] should be corrected so that it reads $\frac{X^1}{X^0} = -\tau$. This leads to some sign changes in other equations there.

[7] As we will see, this differs by an interchange of some components from the standard basis used in $\mathcal{N}=4$ compactifications.

where (p^I, q_I) are the charges of the entire black hole. This provides one real constraint on the extra modulus X^0_∞. As the above equation is invariant under $X^I \to \lambda X^I$ for real X^I, as well as under $X^I \to -X^I$, we see that the magnitude of X^0 is undetermined by this condition, while the phase is determined (in terms of the $X^I, I = 1, 2, 3$) upto a two-fold ambiguity. Another real constraint is now imposed in the form of a "gauge condition":

$$X^I \bar{F}_I - \bar{X}^I F_I = -i \tag{5.4}$$

This determines the magnitude of X^0 but leaves intact the two-fold ambiguity in the phase. The remaining $N-1$ equations then provide constraints on the $\binom{N}{2}$ separations.

For the case $N = 2$ we therefore have a single equation, which completely determines the separation between the two centres. This works as follows. The relevant part of the theory is described by the holomorphic prepotential:

$$F = -\frac{X^1 X^2 X^3}{X^0} \tag{5.5}$$

where the X^I are complex scalar fields related to a subset of the $K3 \times T^2$ moduli, namely $\tau = \tau_1 + i\tau_2$ describing the 2-torus complex structure, and

$$M = \mathrm{diag}(\hat{R}^{-2}, R^{-2}, \hat{R}^2, R^2) \tag{5.6}$$

describing a 2-parameter subset of the remaining moduli (including the $K3$ moduli). The precise relationship is:

$$\frac{X^1}{X^0} = -\tau, \quad \frac{X^2}{X^0} = iR\hat{R}, \quad \frac{X^3}{X^0} = i\frac{\hat{R}}{R} \tag{5.7}$$

The gauge condition eq. (5.4) then tells us that:

$$|X^0_\infty|^2 = \frac{1}{8\hat{R}^2\,\tau_2} \tag{5.8}$$

As in the previous sections, we will consider a dyon with charges $(\vec{Q}, \vec{P})$, but now each taken to be 4-component (the first two components should be thought of as two of the six $\vec{Q}'$ and the second two components constitute two of the six $\vec{Q}''$. The charges correspond to unit torsion, namely:

$$\text{g.c.d.}(Q_i P_j - P_i Q_j) = 1 \tag{5.9}$$

We begin by determining the modulus X^0 in terms of the T-duality invariants $P \circ P, Q \circ Q, P \circ Q$, where as before the inner products are defined in terms of the moduli at infinity, e.g. $P \circ P = P^T (L + M) P$.

As promised, we will use the transcription between the natural electric-magnetic basis $\vec{P}, \vec{Q}$ for the type IIB superstring and the natural basis p^I, q_I for special geometry (see for example ref. [5]):

$$q_I = (Q_1, P_1, Q_4, Q_2), \quad p^I = (P_3, -Q_3, P_2, P_4) \tag{5.10}$$

JHEP12(2008)056

In addition we have:

$$\begin{aligned}
\mathrm{Im}\,(F_0) &= \hat{R}^2\,\mathrm{Im}\,(X^0\tau), \quad &\mathrm{Im}\,(F_1) &= \hat{R}^2\,\mathrm{Im}\,(X^0), \\
\mathrm{Im}\,(F_2) &= \frac{\hat{R}}{R}\,\mathrm{Re}\,(X^0\tau), \quad &\mathrm{Im}\,(F_3) &= R\hat{R}\,\mathrm{Re}\,(X^0\tau)
\end{aligned} \tag{5.11}$$

while

$$\begin{aligned}
\mathrm{Im}\,(X^0) &= \mathrm{Im}\,(X^0), \quad &\mathrm{Im}\,(X^1) &= -\,\mathrm{Im}\,(X^0\tau), \\
\mathrm{Im}\,(X^2) &= R\hat{R}\,\mathrm{Re}\,(X^0), \quad &\mathrm{Im}\,(X^3) &= \frac{\hat{R}}{R}\,\mathrm{Re}\,(X^0)
\end{aligned} \tag{5.12}$$

Inserting these into eqs. (5.3), (5.8), one finds:

$$X^0 = \frac{1}{(2\sqrt{2}\hat{R}\tau_2)}\frac{\sqrt{\Delta}\bar{\tau} + i\,(Q\circ P\bar{\tau} - Q\circ Q)}{\sqrt{Q\circ Q}\,M_{BPS}} \tag{5.13}$$

where M_{BPS} is the BPS mass given by eq. (2.4).

Now let us assume our dyon has n centres of charges $(\vec{Q}^{(i)}, \vec{P}^{(i)})$:

$$\begin{pmatrix} \vec{Q}^{(i)} \\ \vec{P}^{(i)} \end{pmatrix} = \begin{pmatrix} m_i & r_i \\ s_i & n_i \end{pmatrix}\begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix}, \quad i = 1, 2, \cdots, n \tag{5.14}$$

with m_i, r_i, s_i, n_1 integers satisfying:

$$\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} n_i = 1, \quad \sum_{i=1}^{n} r_i = \sum_{i=1}^{n} s_i = 0 \tag{5.15}$$

From eq. (5.10) we find that the charges of the decay products in the q_I, p^J basis are given by:

$$\begin{aligned}
q_I^{(i)} &= (m_iQ_1 + r_iP_1, s_iQ_1 + n_iP_1, m_iQ_4 + r_iP_4, m_iQ_2 + r_iP_2) \\
p^{(i)I} &= (s_iQ_3 + n_iP_3, -(m_iQ_3 + r_iP_3), s_iQ_2 + n_iP_2, s_iQ_4 + n_iP_4)
\end{aligned} \tag{5.16}$$

Now the first term in eq. (5.2) can be written:

$$p^{(i)I}\,\mathrm{Im}\,(F_I) - q_I^{(i)}\,\mathrm{Im}\,(X^I) = \hat{R}\,\mathrm{Re}\,\left(-X^0\ \ X^0\tau\right)\begin{pmatrix} m_i & r_i \\ s_i & n_i \end{pmatrix}\begin{pmatrix} \frac{Q_2}{R} + RQ_4 - i(\frac{Q_1}{\hat{R}} + \hat{R}Q_3) \\ \frac{P_2}{R} + RP_4 - i(\frac{P_1}{\hat{R}} + \hat{R}P_3) \end{pmatrix} \tag{5.17}$$

The invariants $P\circ P, Q\circ Q, Q\circ P$ are given by:

$$\begin{aligned}
Q\circ Q &= \left(\frac{Q_1}{\hat{R}} + \hat{R}Q_3\right)^2 + \left(\frac{Q_2}{R} + RQ_4\right)^2 \\
P\circ P &= \left(\frac{P_1}{\hat{R}} + \hat{R}P_3\right)^2 + \left(\frac{P_2}{R} + RP_4\right)^2 \\
Q\circ P &= \left(\frac{Q_1}{\hat{R}} + \hat{R}Q_3\right)\left(\frac{P_1}{\hat{R}} + \hat{R}P_3\right) + \left(\frac{Q_2}{R} + RQ_4\right)\left(\frac{P_2}{R} + RP_4\right)
\end{aligned} \tag{5.18}$$

The column vector in eq. (5.17) depends on four combinations of Q_i, P_i and therefore cannot in general be expressed in terms of T-duality invariants. Therefore we restrict to the special case, discussed in particular in ref. [5], for which $Q_1 = Q_3 = 0$. In this case only three independent combinations appear in the column vector and it is easy to show that:

$$\begin{aligned} p^{(i)I}\,\mathrm{Im}\,(F_I) - q_I^{(i)}\,\mathrm{Im}\,(X^I) &= \hat{R}\,\mathrm{Re}\,X^0 \begin{pmatrix} -1 & \tau \end{pmatrix} \begin{pmatrix} m_i & r_i \\ s_i & n_i \end{pmatrix} \begin{pmatrix} \sqrt{Q\circ Q} \\ \frac{Q\circ P + i\sqrt{\Delta}}{\sqrt{Q\circ Q}} \end{pmatrix} \\ &= \frac{s_1\sqrt{\Delta}}{2\sqrt{2}\,\tau_2\,M_{BPS}}\;\mathcal{C}(m_i, r_i, s_i, n_i) \end{aligned} \tag{5.19}$$

where $\mathcal{C}(m_i, r_i, s_i, n_i)$ is the curve of marginal stability for multiparticle decays, defined in eq. (4.2).

The numerator of the second term in eq. (5.2), denoted:

$$\mathcal{J}_{ij} \;\equiv\; p^{(i)I} q_I^{(j)} - p^{(j)I} q_I^{(i)} \tag{5.20}$$

is the angular momentum between each pair of decay products evaluated in the moduli-independent norm. We will denote the pairwise separation between the centres by:

$$L_{ij} = |\vec{r}_i - \vec{r}_j| \tag{5.21}$$

Note that $\mathcal{J}_{ij} = -\mathcal{J}_{ji}$ and $L_{ij} = L_{ji}$.

Inserting the above results into eq. (5.2), one finds that it can be expressed as follows:

$$\widetilde{\mathcal{C}}_i + \sum_{j\neq i} \frac{\mathcal{J}_{ij}}{L_{ij}} = 0 \tag{5.22}$$

where

$$\widetilde{\mathcal{C}}_i \equiv \frac{s_i\,\sqrt{\Delta}}{\sqrt{2}\,\tau_2\,M_{BPS}}\;\mathcal{C}(m_i, r_i, s_i, n_i) \tag{5.23}$$

Clearly the first term in eq. (5.22) depends only on the charges of a single centre (as well as the initial charges) while the second term depends on the charges of a pair of centres. Note that we have $\sum_i \widetilde{\mathcal{C}}_i = 0$. Thus we have shown that the curves of marginal stability for multi-centred decays appear also from considerations of multi-centred black holes and the constraints on the locations of their centres.

In the special case considered previously [5, 6] where the dyon has two $\frac{1}{2}$-BPS centres, the corresponding curve of marginal stability is of codimension 1. In this case it is known that the degeneracy of states jumps as we cross the curve. From the supergravity point of view, it was suggested in the $\mathcal{N} = 2$ context in ref. [7] and shown more explicitly in the present $\mathcal{N} = 4$ context in refs. [5, 6], that this decay occurs as a result of the two centres flying apart to infinity at a curve of marginal stability. This is seen by specialising eq. (5.22) to this case. As long as $\mathcal{J}_{12} \neq 0$, the separation $L_{12} \to \infty$ when $\widetilde{\mathcal{C}}_i \to 0$. Moreover for a fixed sign of $\mathcal{J}_{12}$, the separation L_{12} can be positive only on one side of the curve of marginal stability. On the other side it is negative, which indicates that the corresponding two-centred black hole does not exist.

JHEP12(2008)056

Now let us return to the more general case where there are two centres but both are $\frac{1}{4}$-BPS. As we have seen, in this case the locus of marginal stability is not a wall in moduli space, but rather a curve of codimension ≥ 2. Therefore the degeneracy (index) formula cannot jump as one crosses the curve. Hence one need not have expected any relationship between marginal decays and multi-centred dyons. Nevertheless, we see that eq. (5.22) continues to hold in the more general case (with the limitation that the charges are those that can be embedded in an $\mathcal{N} = 2$ compactification).

We interpret this as evidence that the relationship between dyon decay and the disintegration of multi-centred black holes holds more generally than required by the degeneracy formula. Therefore we conjecture that even with the most general charges, n-centred black holes exist in $\mathcal{N} = 4$ string compactifications with generic $\frac{1}{4}$-BPS centres for which eq. (5.22) holds true. It would be worth trying to prove that this is the case, or else to show that such solutions do not exist beyond the cases that can be embedded in the charge space and moduli space of $\mathcal{N} = 2$. An intermediate possibility also exists: that in $\mathcal{N} = 4$ compactifications such multi-centred black holes do exist with arbitrary charges, but only on a subspace of the moduli space.

Examining eq. (5.22) one sees that if the marginal stability condition $\widetilde{\mathcal{C}}_i = 0$ is satisfied for a particular i, then we must have:

$$\sum_{j \neq i} \frac{\mathcal{J}_{ij}}{L_{ij}} = 0 \tag{5.24}$$

One possible solution is to have $L_{ij} \to \infty$ for all $j \neq i$. This means the ith centre has been taken infinitely far away from all the others, in agreement with the picture of marginal decay that we developed in section 4 above. Since the pairwise Saha angular momenta $\mathcal{J}_{ij} \equiv P^{(i)} \cdot Q^{(j)} - P^{(j)} \cdot Q^{(i)}$ cannot all be positive in every equation (since $\mathcal{J}_{ij} = -\mathcal{J}_{ji}$) there could be other configurations where the $\widetilde{\mathcal{C}}_i = 0$, except in the case of two centres. It is not clear to us how these other solutions should be interpreted.

Note that in the above equation the angular momentum is measured with respect to moduli-independent inner product $P \cdot Q \equiv P^T L Q$ unlike the angular momentum appearing in the curve of marginal stability eq. (4.2) which is computed using the moduli-dependent inner product $P \circ Q \equiv P^T (L + M) Q$. One may think of the latter evolving to the former as we follow the attractor flow from infinity to the horizon of the black hole. However it would be nice to have a better physical understanding of the role of dyonic angular momenta in these discussions.[8]

6. Conclusions

In this work we have obtained the loci of marginal stability for decays of $\frac{1}{4}$-BPS dyons into any number of BPS constituents in $\mathcal{N} = 4$ string compactifications. These loci appear as equations constraining the $132 + 2$ moduli, more precisely as a curve of marginal stability

[8]As is well-known, the dyonic angular momentum plays a physical role in the wall-crossing formulae [7, 17, 5, 6, 18, 19] that describe how the degeneracy jumps, but in the present discussions there are no walls or jumps.

in the upper-half-plane that represents a torus moduli space (in the basis of type IIB on $K3 \times T^2$, this is the geometric torus) as well as some more complicated equations on the remaining moduli. While in this paper we worked with unit-torsion initial dyons, it should be quite straightforward to extend our results to general torsion. We showed how to extend our analysis to multi-particle decays, and found a relation between the loci of marginal stability obtained in this way and the supergravity constraints on pairwise separations of the centres of multi-centred black holes.

The physical role of “rare” marginal dyon decays, namely all those other than of a $\frac{1}{4}$-BPS dyon into two $\frac{1}{2}$-BPS dyons, has yet to be explored. Because such decays take place on loci of codimension ≥ 2 in moduli space, they do not form “domain walls” across which the degeneracy can jump. Therefore they do not affect the basic entropy or dyon counting formulae. However it is certainly possible that they have other interesting physical effects which may emerge on further investigation.

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